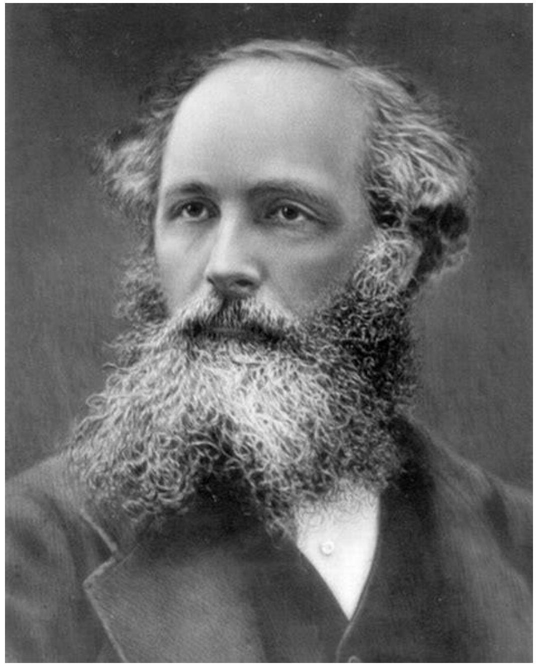
Maxwell’s equations

A BRIEF INTRODUCTION

What are Maxwell’s equations?

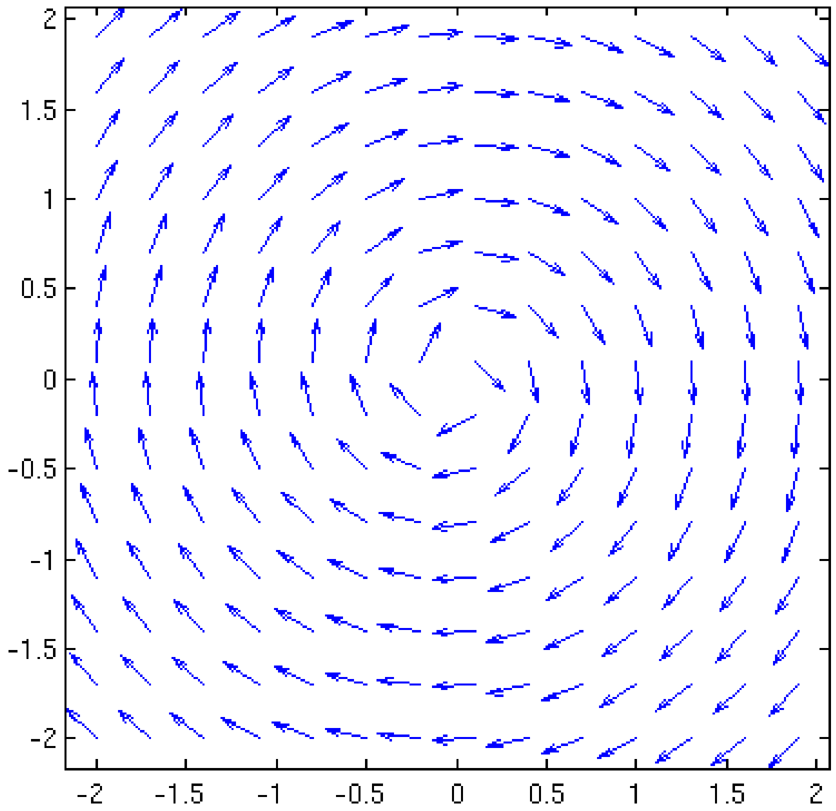
- A set of 4 equations which govern the physics of electromagnetism
- They describe how electric and magnetic fields propagate and interact with each other and with other objects, in classical mechanics
- James Clerk Maxwell (1831-1879) built on the previous work of other great minds, such as Faraday and Gauss, to produce these equations
- They unify and generalise the preceding experimental laws into 4 concise equations with a solid theoretical foundation
- The speed of light in a vacuum can be determined from the equations, which led to the realisation that light is an electromagnetic wave



Note that the equations in this presentation are in differential form and in the absence of magnetic or polarizable materials

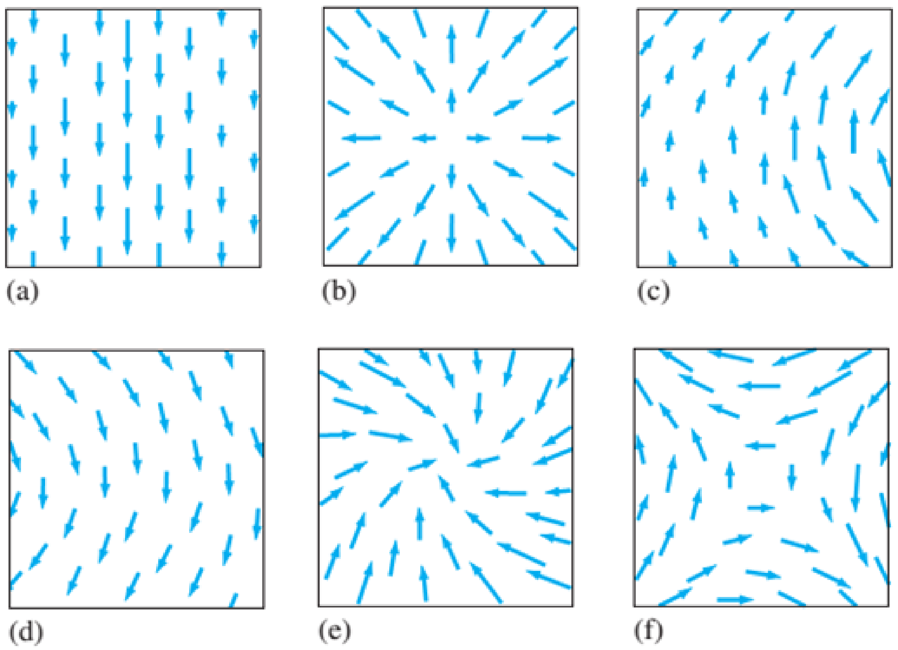
Vector fields

- Fields can be represented by a vector at every point in space
- The direction of the vector represents the direction of the force and the length represents the magnitude of the force at that point in space
- A graphical representation would use arrows at regular intervals through space, like so:
- Vector fields can be used to represent all sorts of other situations - anywhere where a vector can describe each point in space
- Another example is the wind direction and speed on a weather map



Divergence and curl

- Divergence is essentially a measure of how much a vector field is directed away from a point
- Curl is essentially a measure of how much a vector field is directed around a point
- They can be negative, positive or zero
- Divergence of a vector field $\boldsymbol{F}$ is denoted by $div\ \boldsymbol{F}$ or $\nabla \cdot \boldsymbol{F}$
- Curl of a vector field $\boldsymbol{F}$ is denoted by $curl\ \boldsymbol{F}$ or $\nabla \times \boldsymbol{F}$



The four equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

Generalisation of Gauss' Law for Electricity

- The divergence and hence electric flux of the electric field out of any area is directly proportional to the charge within that area
- The electric field points from positive to negative charge
- In a vacuum, $\rho = 0$ (there are no charges), so $\nabla \cdot \boldsymbol{E} = 0$; thus an electric field cannot be produced from nothing

$\nabla \cdot \boldsymbol{E}$ is the divergence of the electric field

ρ is the charge density of a given region

ε_0 is the electric permittivity of free space

$$\nabla \cdot \boldsymbol{B} = 0$$

Generalisation of Gauss' Law for Magnetism

- The divergence and hence magnetic flux of the magnetic field at any point in space is zero – magnetic flux cannot appear or disappear
- The magnetic flux out of the north pole equals the magnetic flux into the south pole, which prevents the existence of monopoles – magnets with one pole
- The magnetic field points from north to south

$\nabla \cdot \boldsymbol{B}$ is the divergence of the magnetic field

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Generalisation of Faraday's Law of Induction or Lenz’s Law

- A magnetic field that varies with time will produce an electric field that varies in space
- The voltage around a circuit is equal to the negative of the rate of change of the magnetic field through the area enclosed by the circuit; a voltage is induced in a circuit when it is within a changing magnetic field
- This is the basis for electric generators, inductors and transformers

$\nabla \times \boldsymbol{E}$ is the curl of the electric field

$\frac{\partial \boldsymbol{B}}{\partial t}$ is the rate of change of the magnetic field over time

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

Generalisation of Ampere's Law

- An electric field that varies with time will produce a magnetic field that varies with space
- Current flowing through a wire will produce a magnetic field around it
- In a vacuum, $\boldsymbol{J} = 0$, so the equation is essentially of the same form as the previous equation (just with different constants)

$\nabla \times \boldsymbol{B}$ is the curl of the magnetic field

μ_0 is the magnetic permeability of free space

$\boldsymbol{J}$ is the current density (current per unit area)

ε_0 is the electric permittivity of free space

$\frac{\partial \boldsymbol{B}}{\partial t}$ is the rate of change of the magnetic field over time

Electromagnetic constants

- Permeability is, in a simple sense, how well a material supports the formation of a magnetic field within itself. Permittivity is similar but applies to the electric field.
- The speed of light is related to permittivity and permeability through the relationship $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$
- Therefore, the speed of light is intrinsically related to Maxwell's equations: for example, his fourth equation can be written as $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}$

Permeability of free space $\mu_0 = 1.26 \times 10^{-6}\ NA^{-2}$
Permittivity of free space $\varepsilon_0 = 8.85 \times 10^{-12}\ Fm^{-1}$
Speed of light $c = 3 \times 10^{8}\ ms^{-1}$